Exercise 28

Prove the statement using the precise definition of a limit.

$$\lim_{x\to 4^+} \frac{2}{\sqrt{x-4}} = \infty$$

Solution

Proving this infinite limit is logically equivalent to proving that

$$\text{if} \quad 4 < x < 4+\delta \qquad \text{then} \qquad \frac{2}{\sqrt{x-4}} > M,$$

where M is any positive number. Start by working backwards, looking for a number δ that satisfies $4 < x < 4+\delta$, or $0 < x-4 < \delta$.

$$\frac{2}{\sqrt{x-4}} > M$$

$$2 > M\sqrt{x-4}$$

$$\frac{2}{M} > \sqrt{x-4}$$

$$\left(\frac{2}{M}\right)^2 > \left(\sqrt{x-4}\right)^2$$

$$\frac{4}{M^2} > x-4$$

Choose $\delta = \frac{4}{M^2}$. The hypothesis then becomes

$$4 < x < 4+\delta$$

$$0 < x-4 < \delta$$

$$x-4 < \frac{4}{M^2}$$

$$M^2(x-4) < 4$$

$$M^2 < \frac{4}{x-4}$$

$$\sqrt{M^2} < \sqrt{\frac{4}{x-4}}$$

$$M < \frac{2}{\sqrt{x-4}}.$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 4^+} \frac{2}{\sqrt{x-4}} = \infty.$$